Lecture Series on Lyapunov Exponents

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1 Basic Notions

1.1 Introduction

Let us begin with an easy problem. Let $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The spectrum of M consisting of its eigenvalues is given by $\sigma(M) = \left\{\tau, -\tau^{-1}\right\}$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. We now have the following question.

Question 1.1.1. What is the value of

$$\chi = \lim_{n\to\infty} \frac{1}{n} \log \|M^n\|?$$

Intuitively, one guesses that the answer must be $\chi = \log(\tau)$, with τ being the spectral radius of M, which dictates the dominant exponential behavior of M^n. One can even prove this easily since M is diagonalisable and one can instead consider the product of the diagonal matrices containing the eigenvalues.

Now, let us make it *slightly* difficult by looking at two matrices instead of one. Consider the matrices $M_1 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$, whose respective spectra are given by $\sigma(M_1) = \left\{\tau^2, -\tau^{-2}\right\}$ and $\sigma(M_2) = \left\{2 \pm \sqrt{3}\right\}$. Define $M^{(n)}$ to be a random product (say both matrices have equal probability $p = \frac{1}{2}$) of n matrices from the set $\left\{M_1, M_2\right\}$, which one can formally write as

$$M^{(n)} = M_{i_{n-1}} M_{i_{n-2}} \cdots M_{i_1} M_{i_0}$$

where $i_j \in \left\{1, 2\right\}$.

Question 1.1.2. What is the *generic* value of

$$\chi = \lim_{n\to\infty} \frac{1}{n} \log \|M^{(n)}\|?$$

Does it even exist?

Indeed, this limit exists for a.e. sequence $(i_j)_{j\in\mathbb{N}_0}$. One might guess that it is just given by the average of $\log(\lambda_1)$ and $\log(\lambda_2)$. Surprisingly, this is not the case, and it has been calculated that

$$\chi = 1.143311035 \neq \frac{1}{2}\big(\log(\tau^2) + \log(2+\sqrt{3})\big) \approx 1.13969077.$$

The complete derivation for this example, and for a general family of matrices can be found in [Pol10]. In Kingman's seminal paper [Kin73], he said that "pride of place among the unsolved problems of subadditive ergodic theory must go to the calculation of the number χ", which gives a natural answer to the question, "why are Lyapunov exponents interesting?"; they are interesting because they are hard to compute and there is no general way of calculating them.

1.2 Lyapunov Exponents

For the following review of basic material, we use [Via13] and [BP07] as our main reference texts.

Definition 1.2.1. Given a sequence $\{M_j\}_{j\geqslant 0}$ of matrices in $\mathrm{Mat}(d,\mathbb{C})$, satisfying the condition

$$\limsup_{m\to\infty} \frac{1}{m} \log \|M^{(m)}\| < \infty, \tag{1.1}$$

one can consider its *Lyapunov exponent* $\chi : \mathbb{C}^d \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\chi(v) = \limsup_{n\to\infty} \frac{1}{n} \log \|M^{(n)} v\|, \tag{1.2}$$

where we have set $M^{(n)} := M_{n-1}M_{n-2}\cdots M_1 M_0$.

The function χ satisfies the following properties.

(1) $\chi(\alpha v) = \chi(v)$, for all $v \in \mathbb{C}^d$, $\alpha \in \mathbb{R}\setminus\{0\}$

(2) $\chi(v+w) \leqslant \max\{\chi(v), \chi(w)\}$, for all $v, w \in \mathbb{C}^d$

(3) $\chi(0) = -\infty$.

Here, we follow the convention that $\log(0) = -\infty$. We also note that $\chi(v)$ does not depend on the norm $\|\cdot\|$ chosen as they are all equivalent in the finite-dimensional case.

Remark 1.2.2. The property given in Eq. 1.1 ensures that $\chi(v) < \infty$. This is immediately satisfied when $\sup_j \|M_j\| < \infty$ or when $\lim_{n\to\infty} \frac{1}{n} \log^+ \|M_j\| < \infty$. $\Diamond$

Example 1.2.3. Let X be an r-dimensional smooth submanifold of $\mathbb{R}^d$ and $g : X \to X$ be a C^1-diffeomorphism of X. Let $x \in \mathbb{X}$, T_xX be the tangent space at x, and $v \in T_xX$. The Lyapunov exponent associated to x is given by

$$\chi(x, v) = \lim_{n\to\infty} \frac{1}{n} \log \|Dg^n(x)v\|,$$

where $Dg^n(x)$ is the Jacobi matrix of g^n at x. $\Diamond$

Example 1.2.4 ([Bar17, Ex. 1.1.2])**.** Let $A : \mathbb{R}_{\geqslant 0} \to \mathrm{Mat}(d,\mathbb{R})$. The Lyapunov exponent χ associated with the differential equation $v' = A(t)v$ is given by

$$\chi(v_0) = \limsup_{t\to\infty} \frac{1}{t} \log \|v(t)\|$$

where $v(t)$ is the solution of the IVP

$$v' = A(t)v \quad \text{ and } v(0) = v_0.$$

$\Diamond$

1.3 Properties

Proposition 1.3.1. *Let χ be the Lyapunov exponent defined above.*

(1) *If $\chi(v) \neq \chi(w)$, then $\chi(v+w) = \max\{\chi(v), \chi(w)\}$*

(2) *If $v_1, \dots, v_m \in \mathbb{C}^d$, $\alpha_1, \dots, \alpha_m \in \mathbb{R} \setminus \{0\}$, then*

$$\chi(\alpha_1 v_1 + \cdots + \alpha_m v_m) = \max\{\chi(v_i) \mid 1 \leqslant i \leqslant m\}.$$

(3) *If for some $v_1, \dots, v_m \in \mathbb{C}^d$, the values $\chi(v_1), \dots, \chi(v_m)$ are distinct, then $v_1, \dots, v_m$ are linearly independent.*

(4) *χ attains at most d distinct finite values*

Proof.

(1) Suppose $\chi(v) < \chi(w)$. One then has

$$\chi(v+w) \leqslant \chi(w) = \chi(v+w-v) \leqslant \max\{\chi(v+w), \chi(v)\}$$

If $\chi(v+w) < \chi(v)$, then $\chi(w) \leqslant \chi(v)$, which is a contradiction. Hence, $\chi(v+w) \geqslant \chi(v)$, implying $\chi(v+w) = \chi(w) = \max\{\chi(v), \chi(w)\}$.

(2) Follows from Properties 1 and 2, and (1).

(3) Assume on the contrary that $v_1, \dots, v_m$ are linearly dependent, i.e.,

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0,$$

with not all $\alpha_i = 0$, while $\chi(v_1), \dots, \chi(v_m)$ are distinct. This yields

$$\chi(\alpha_1 v_1 + \cdots + \alpha_m v_m) = -\infty = \max\{\chi(v_i) \mid 1 \leqslant i \leqslant m,\} \neq -\infty,$$

which is a contradiction.

(4) Follows from (3)

□

1.4 Filtrations

From the distinct values $\chi_1, \dots, \chi_{d'}$, one can construct a *filtration* of $\mathbb{C}^d$, i.e., a nested sequence of subspaces $\{\mathcal{V}\}_{i=1}^{d'}$

$$\mathbb{C}^d = \mathcal{V}^1 \supsetneq \mathcal{V}^2 \supsetneq \ldots \supsetneq \mathcal{V}^{d'} \neq \{0\} \tag{1.3}$$

such that $\chi(v) = \chi_i$, for all $v \in \mathcal{V}^i \setminus \mathcal{V}^{i+1}$.

Remark 1.4.1. In the case where $\{M_j\}_{j \geqslant 0}$ is made up of a single matrix M, the Lyapunov exponents χ_i are given by $\log|\lambda_i|$, where λ_i are the eigenvalues of M and, $\mathcal{V}^i \setminus \mathcal{V}^{i+1}$ are the corresponding (possibly generalised) eigenspaces. When $\{M_j\}_{j \geqslant 0}$ is a convergent sequence with limit M, the values of the exponents are also determined by the eigenvalues of M. ◊

1.5 Regularity

Definition 1.5.1. A sequence $\{M_j\}_{j\geqslant 0}$ is said to be *forward regular* if

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\det\left(M^{(n)}\right)\right| = \sum_{i=1}^{d}\chi_i', \tag{1.4}$$

provided that the limit exists.

Here, $\chi_1' \geqslant \ldots \geqslant \chi_d'$ are the values attained by χ, counted with their multiplicities. Mere existence of the limit does not guarantee forward regularity, as we shall see in the following example.

Example 1.5.2. Let $M_0 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ and $M_j = \begin{pmatrix} 1 & 0 \\ -2^{j+1} & 4 \end{pmatrix}$, for each $j \geqslant 1$, which makes $M^{(n)} = \begin{pmatrix} 1 & 0 \\ 2^n & 4^n \end{pmatrix}$. A direct calculation gives $\det M^{(n)} = 4^n$, which yields

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\det\left(M^{(n)}\right)\right| = 2\log(2)$$

However, it is easy to check that for $v_1 = (1,0)^{\mathrm{T}}$ and $v_2 = (0,1)^{\mathrm{T}}$, one has

$$\chi(v_1) = \log(2) \quad \text{and} \quad \chi(v_2) = 2\log(2),$$

and hence

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\det\left(M^{(n)}\right)\right| = 2\log(2) \neq \chi_1 + \chi_2 = 3\log(2).$$

◊

1.6 Matrix cocycles

Let (X,T) be a topological dynamical system. A measure μ on X is invariant with respect to T if $\mu(\mathcal{E}) = \mu(T^{-1}(\mathcal{E}))$, for all Borel sets $\mathcal{E}$. With the invariant measure μ, (X,T,μ) becomes a measure-theoretic (or metric) dynamical system.

A transformation T is *ergodic* with respect to μ, if $T^{-1}(\mathcal{E}) = \mathcal{E}$ implies either $\mu(\mathcal{E}) = 0$ or $\mu(\mathcal{E}) = 1$. Analogously, μ is an ergodic measure for the topological dynamical system (X,T) if the same condition is satisfied.

Example 1.6.1 (Irrational rotations)**.** Let $X = [0,1) \simeq S^1$ and T be a rotation given by $T_\alpha : x + \alpha$, for some $\alpha \in [0,1)$. The map T_α is ergodic with respect to the Haar measure on X (which is the Lebesgue measure) if and only if α is irrational.

Theorem 1.6.2 (Birkhoff's Ergodic Theorem)**.** *Let (X,T,μ) be an ergodic measure-preserving dynamical system, $g \in L^1(\mu)$. Then, one has*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} g(T^j x) = \int_X g(\xi)\,\mathrm{d}\mu(\xi),$$

for μ-a.e. $x \in X$. □

One way to generate sequences of matrices is via cocycles. Consider a measure-preserving dynamical system (X, f, μ) and a measurable matrix-valued map $A : X \to \mathrm{Mat}(d, \mathbb{C})$.

Definition 1.6.3. A skew linear map $F : X \times \mathbb{C}^d \to X \times \mathbb{C}^d$ defined by $(x, v) \mapsto \big(f(x), A(x)v\big)$ is called a *linear cocycle over* f, where f is the *base dynamics* of the cocycle.

We call F *ergodic* over (X, f, μ) if f is ergodic with respect to μ. An iteration of this function yields the pair $F^n(x, v) = (f^n(x), A^{(n)}(x)v)$, where the induced fibre action on $\mathbb{C}^d$ is determined by the matrix product

$$A^{(n)}(x) = A(f^{n-1}(x)) \cdot \ldots \cdot A(f(x))A(x).$$

Unless otherwise stated, we assume the base dynamics to be fixed, and we refer to $A^{(n)}(k)$ as the matrix cocycle.

Example 1.6.4.

(1) Let $\Omega \subset \mathrm{Mat}(d, \mathbb{C})$ be compact. Let $X = \Omega^{\mathbb{Z}}$ with the (left-sided) shift operator S on X, with $(Sx)_k = x_{k+1}$, for $\{x_k\}_{k\in\mathbb{Z}} \in \Omega^{\mathbb{Z}}$, and μ a probability measure on Ω. Consider the locally constant map $A : x = (\ldots, x_{-1}, x_0, x_1, \ldots) \mapsto A(x_0)$. Then, (S, A) defines a cocycle over $X \times \mathbb{C}^d$. Furthermore, S is ergodic with respect to the product measure $\mu^{\mathbb{Z}}$.

(2) Let $X = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $A : \mathbb{T}^d \to \mathrm{Mat}(d, \mathbb{C})$, and $\widetilde{M}$ be a *toral endomorphism* given by $\widetilde{M} : x \mapsto (Mx) \mod 1$, where $M \in \mathrm{Mat}(d, \mathbb{Z})$. It is well known that $\widetilde{M}$ is ergodic with respect to Lebesgue measure whenever $\det M \neq 0$ and M does not have eigenvalues which are roots of unity [EW11, Cor. 2.20], and is invertible whenever $M \in \mathrm{GL}(d, \mathbb{Z})$, i.e., $\det M = \pm 1$. As in the first example, $(\widetilde{M}, A)$ defines a matrix cocycle. ◊

For sequences arising from cocycles, more specific versions of Eq. (1.2) and Eq. (1.3) for the Lyapunov exponent $\chi : \mathbb{C}^d \times X \to \mathbb{R} \cup \{-\infty\}$ and the x-dependent filtration it defines read

$$\chi(v, x) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)v\| \qquad \text{and} \qquad \mathbb{C}^d = \mathcal{V}_x^1 \supsetneq \mathcal{V}_x^2 \supsetneq \ldots \supsetneq \mathcal{V}_x^{d'(x)} \neq \{0\}\,,$$

with $\chi(v, x) = \chi_i(x)$, for all $v \in \mathcal{V}_x^i \setminus \mathcal{V}_x^{i+1}$. We say that $A^{(n)}(x)$ at a given point x is forward regular if the sequence $\{A(f^n(x))\}_{n\geqslant 0}$ is forward regular.

Lemma 1.6.5. *Let $v \in \mathbb{C}^d \setminus \{0\}\,, x \in X$. Assuming $A^{(n)}(x)^{-1}$ exists, one has,*

$$\chi_{\min}(x) \leqslant \chi(x, v) \leqslant \chi_{\max}(x),$$

where

$$\chi_{\max}(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)\| \quad \textit{and} \quad \chi_{\min}(x) = \liminf_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)^{-1}\|^{-1}.$$

Proof. Note that the following holds for all non-zero v,

$$\|A^{(n)}(x)^{-1}\|^{-1}\|v\| \leqslant \|A^{(n)}(x)v\| \leqslant \|A^{(n)}(x)\|\|v\|.$$

The claim then directly follows by taking the logarithm, and the lim sup and the lim inf of the upper and the lower bound, respectively. □

Define $\phi^+(x) := \max\{0, \phi(x)\}$. The following result on the extremal exponents is due to Furstenberg and Kesten [FK60]; see also [Via13, Thm. 3.12].

Theorem 1.6.6 (Furstenberg–Kesten)**.** *Let $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ be a matrix cocycle defined by $F(x, v) = (f(x), A(x)v)$, where $A : X \to \mathrm{GL}(d, \mathbb{R})$ is measurable, and X is compact. If $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$, the extremal exponents $\chi_{\min}(x)$ and $\chi_{\max}(x)$ exist as limits for a.e. $x \in X$. Moreover, these functions are f-invariant and are μ-integrable.* $\square$

2 Main Theorems and Central Results

The following generalisation of Birkhoff's ergodic theorem for subadditive functions is due to Kingman [Kin73]; compare [Via13, Thm. 3.3].

Theorem 2.0.1 (Kingman's subadditive ergodic theorem)**.** *Let X be a compact space and assume $f : X \to X$ to be invariant with respect to μ. Let $\{\phi_n\}$ be a sequence of functions such that $\phi_1^+ \in L^1(\mu)$ and*

$$\phi_{m+n} \leqslant \phi_m + \phi_n \circ f^m \qquad \text{holds for all } m, n \geqslant 1. \tag{2.1}$$

Then, $(\frac{\phi_n}{n})$ converges μ-a.e. to some $\phi : X \to [-\infty, \infty)$. Moreover, ϕ^+ is integrable and

$$\int_X \phi(\xi)\mathrm{d}\mu(\xi) = \lim_{n\to\infty} \frac{1}{n} \int_X \phi_n(\xi)\mathrm{d}\mu(\xi) = \inf_n \frac{1}{n} \int_X \phi_n(\xi)\mathrm{d}\mu(\xi) \in [-\infty, \infty).$$

Furthermore, when f is ergodic, ϕ is constant for μ-a.e. $x \in X$. □

Example 2.0.2. A relevant class of functions satisfying Eq. 2.1 are those arising from cocycles, which are of the form $\phi_m = \log \|A^{(m)}(x)\|$, where $\|\cdot\|$ is a submultiplicative matrix norm. One can explicitly see this by looking at the decomposition of $A^{(m+n)}$ given by

$$A^{(m+n)}(x) = \underbrace{A(f^{m+n-1}x)\cdots A(f^m x)}_{A^{(n)}(f^m x)} \cdot \underbrace{A(f^{m-1}x)\cdots A(fx)A(x)}_{A^{(m)}(x)}.$$

◊

Example 2.0.3. Not all matrix norms are submultiplicative. An example of a nonsubmultiplicative norm is

$$\|M\|_{\max} = \max_{i,j} |m_{ij}|$$

One can easily check for $M = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ that this is not submultiplicative, i.e.,

$$\|M^2\|_{\max} = 2 > 1 = \|M\|_{\max}\|M\|_{\max}.$$

◊

For ergodic real-valued cocycles, one has the following central result in the theory of Lyapunov exponents, which is due to Oseledec [Ose68]; see [Via13, Thm. 4.1] and [BP07, Thm. 3.4.3].

Theorem 2.0.4 (Oseledec's multiplicative ergodic theorem, one-sided)**.** *Let f be an ergodic transformation on a compact space X with an invariant measure μ. Let $A : X \to \mathrm{GL}(d, \mathbb{R})$ be measurable, such that the condition $\log^+ \|A\| \in L^1(\mu)$ holds. Then, for μ-a.e. $x \in X$, the cocycle $A^{(n)}(x)$ is forward regular. Moreover, for these x, the Lyapunov exponents $\chi_i(x)$ are constant, i.e., there exist real numbers $\chi_1, \dots, \chi_{d'}$, and a filtration*

$$\mathbb{R}^d = \mathcal{V}_x^1 \supsetneq \mathcal{V}_x^2 \supsetneq \dots \supsetneq \mathcal{V}_x^{d'} \neq \{0\}$$

such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)v_i\| = \chi_i$$

for all $v_i \in \mathcal{V}_x^i \setminus \mathcal{V}_x^{i+1}$. $\square$

There exists an even stronger notion of regularity, also known as *Lyapunov–Perron regularity*. This requires both the matrix-valued function A and the map f to be invertible so that one can define $A^{(n)}(x)$, for $n < 0$. Under these invertibility assumptions, and that $\log^+ \|A^{-1}\| \in L^1(\mu)$, one gets the following two-sided version of Theorem 2.0.4.

Theorem 2.0.5 (Oseledec's multiplicative ergodic theorem, two-sided)**.** *Let f be an* invertible *ergodic transformation on a compact space X with an invariant measure μ. Let $A : X \to$* $\mathrm{GL}(d, \mathbb{R})$ *be measurable, such that the condition* $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$ *holds. Then, for μ-a.e. $x \in X$, the cocycle $A^{(n)}(x)$ is Lyapunov–Perron regular. Moreover, for these x, the Lyapunov exponents $\chi_i(x)$ are constant, i.e., there exist real numbers $\chi_1, \ldots, \chi_{d'}$, and a splitting*

$$\mathbb{R}^d = E_x^1 \oplus E_x^2 \oplus \ldots \oplus E_x^{d'} \neq \{0\}$$

such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^{(n)}(x)v_i\| = \chi_i$$

for all $v_i \in E_x^i$. Furthermore, the subspaces E_x^j are invariant with respect to $A(x)$, i.e., for a.e. x, $A(x)E_x^i = E_{f(x)}^i$. $\square$

3 Lyapunov Exponents in Diffraction Theory of Aperiodic Structures

3.1 Aperiodic structures

Before 1992, the definition of a crystal by the IUCr is a material with a periodic structure, which mathematically corresponds to a lattice in $\mathbb{R}^n$ (where n is usually 2 or 3). This has since been changed to those solids which display pure point diffraction. The theory of aperiodic order deals with structures having no translational symmetry but might possess a certain type of long-range order. In physics, this long-range order is usually characterised by a set of points of high intensity in the diffraction image obtained when one subjects the said structure to a diffraction experiment. These bright points are known as Bragg peaks, and their presence implies that the structure has pure point diffraction component. This area of research was boosted by Schechtman's discovery [SBGC84] of a material with fivefold rotational symmetry, and hence does not possess a lattice structure (could not be indexed using any of the 14 Bravais lattices in three dimensions), but has pure point diffraction; see Fig. 1.

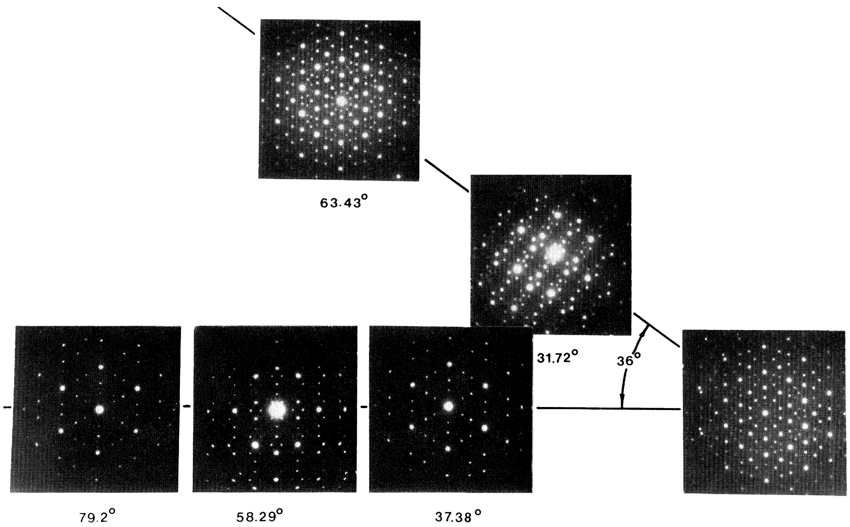


Figure 1: Diffraction image of the AlMn-alloy quasicrystal discovered by Schechtman, taken from [SBGC84]

It is an easy exercise to show that in $\mathbb{R}^2$ and $\mathbb{R}^3$, lattices only admit a limited number of allowable rotational symmetries.

Proposition 3.1.1 ([BG13, Lem. 3.2]). *Let $\varGamma$ be a lattice in $\mathbb{R}^d$, $d \in \{2,3\}$ and $R \in \mathbb{O}(d)$. If $R\varGamma = \varGamma$, then the characteristic polynomial $P(\lambda) = \det(R - \lambda\mathbb{I}) \in \mathbb{Z}[\lambda]$.*

Proof. Let $\{b_1, \ldots, b_d\}$ be a basis for $\varGamma$, so that $\varGamma = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d$. If $R\varGamma = \varGamma$, one then has

$$Rb_i = \sum_j b_j a_{ji}, \quad \text{with } a_{ji} \in \mathbb{Z}.$$

In matrix form, one has $RB = BA$, where B is the basis matrix of $\varGamma$. Since B is invertible, one gets

$$R = BAB^{-1},$$

where $A \in \mathrm{GL}(d, \mathbb{Z})$. The claim follows since A and R share the same characteristic polynomial. □

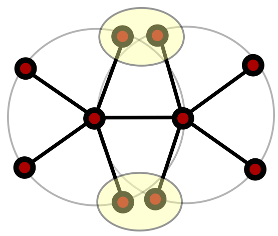


Figure 2: Illustration of the imposibility of five-fold rotational symmetry for lattices

Corollary 3.1.2 ([BG13, Cor. 3.1]). *A lattice $\varGamma \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ can have n-fold rotational symmetry only for $n \in \{1, 2, 3, 4, 6\}$.*

Proof. In dimension $d = 2$, a rotation matrix is given by

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

whose characteristic polynomial is $P(\lambda) = \lambda^2 - 2\cos(\theta) + 1$. If $P(\lambda) \in \mathbb{Z}[\lambda]$, then one has $|\cos(\theta)| \in \{0, \frac{1}{2}, 1\}$, which immediately implies the claim.

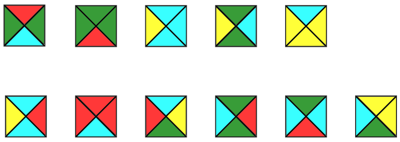
For $d = 3$, one uses the well-known fact that R_ϕ is similar to a matrix of the form

$$R'_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

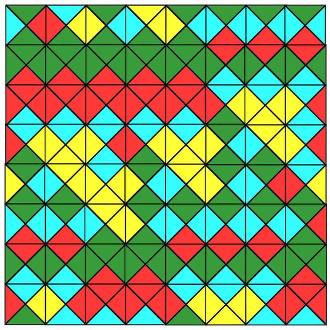
whose characteristic polynomial is $P(\lambda) = (1 - \lambda)(\lambda^2 - 2\cos(\theta) + 1)$, which yields the same conclusion. □

Before Schechtman's discovery, the first paradigm shift from purely periodic structures stemmed from the study of almost periodic functions; see [Boh93, Esc04, Boh47, Bes54]. Notably, notions of Fourier transformability for unbounded measures had already seen reasonable progress by the early '70s; see [AdL74].

Alongside these developments in harmonic analysis was a proliferation of important results on non-periodic tilings. The undecidability of the domino problem was established by Berger in 1966 [Ber66], which meant a tiling of the plane via a finite set of decorated tiles need not be periodic.



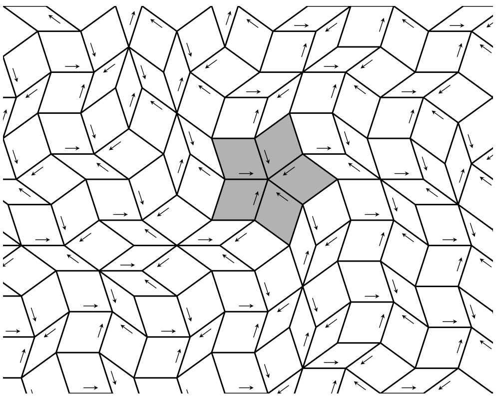
(a) The Jeandel-Rao Wang tiles



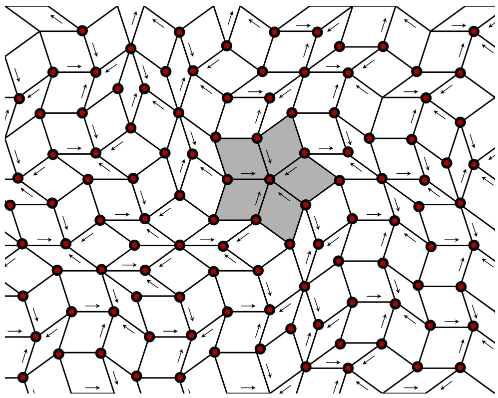
(b) A patch of a Jeandel-Rao tiling

Within a decade, Penrose solved a related but geometrically different problem in his monumental discovery of tilings of $\mathbb{R}^2$ by six prototiles having no translational symmetry (and hence are non-crystallographic) [Pen74].

Although there were lots of works which already applied Fourier analysis on aperiodic tilings, it was the work of Dworkin [Dwo93] and Hof [Hof95] that set the stage for mathematical diffraction of aperiodic structures. Under this formalism, one normally views a vertex set Λ of an aperiodic tiling $\mathcal{T}$ as a model for a quasicrystal; see Fig. 3. One distinguishes different atoms by placing different weights signifying distinct scattering strengths. The non-periodicity of such tilings imply that one must deal with (weighted) unbounded measures to describe atomic positions.



(a) A patch of an aperiodic tiling



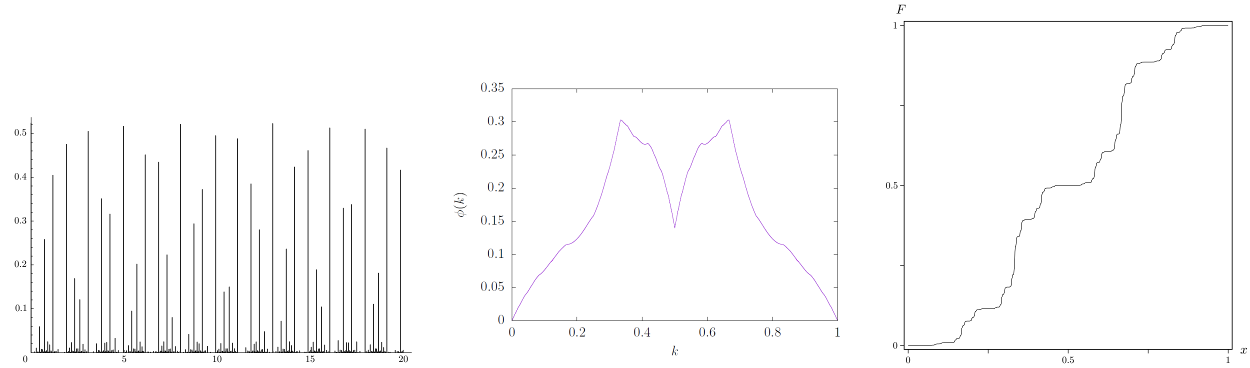
(b) Point sets via vertex points

Figure 3: Aperiodic point sets via tilings

3.2 Diffraction of aperiodic structures

Motto: Mathematical diffraction is Fourier analysis on point sets.

A nice and detailed survey of mathematical diffraction can be found in [BG13, Ch. 9], and a list of works on Lyapunov exponents in the diffraction setting would contain [BFGR19, BGäM18, BGrM18, Man17a, Man19].



(a) $\widehat{\gamma}_{\mathsf{pp}}$ (b) $\widehat{\gamma}_{\mathsf{ac}}$ (c) $\widehat{\gamma}_{\mathsf{sc}}$

Figure 4: Illustration of the three different spectral components

A point set $\varLambda$ in $\mathbb{R}^d$ is a countable union of singletons. We usually require that these point sets ares

UD *Uniformly discrete*: there exists an $R_{\mathrm{c}} > 0$ such that, $B_{R_{\mathrm{c}}}(x) \cap \varLambda = \{x\}$ for all $x \in \varLambda$

RD *Relatively dense*: there exists an $R_{\mathrm{p}} > 0$ such that $B_{R_{\mathrm{p}}}(x)$ contains at least two points in $\varLambda$, for all $x \in \varLambda$.

Point sets satisfying these two properties are called *Delone* sets. From an infinite Delone set $\varLambda \subset \mathbb{R}^d$, one can construct a weighted Dirac comb $\omega = \sum_{x \in \varLambda} w(x)\delta_x$, which represents the atomic configuration, and distinguishes positions occupied by different atom types by different values (typically complex) of the weight function $w(x)$.

Aperiodic tilings naturally give rise to these point sets when one considers their vertex sets. This choice is not unique, as one could also choose a point in the interior of the tile to be the tile's control point.

From ω, one then computes the autocorrelation measure γ via $\gamma = \omega \circledast \widetilde{\omega}$. Then, one takes the Fourier transform of γ to get the diffraction measure $\widehat{\gamma}$. The diffraction $\widehat{\gamma}$ is a positive measure, and hence has the Lebesgue decomposition given by

$$\widehat{\gamma} = \widehat{\gamma}_{\mathsf{pp}} + \widehat{\gamma}_{\mathsf{ac}} + \widehat{\gamma}_{\mathsf{sc}}.$$

Here, $(\widehat{\gamma})_{\mathsf{pp}}$ is the *pure point* component and is the analytic analogue of Bragg peaks in a diffraction experiment, $(\widehat{\gamma})_{\mathsf{ac}}$ is the *absolutely continuous* component represented by a locally-integrable function $h(k)$ whose non-triviality is usually attributed to a certain level of randomness, and in particular, represents what is called diffuse diffraction, and $(\widehat{\gamma})_{\mathsf{sc}}$ is the *singular continuous* component, which lives on an uncountable set of measure zero and is difficult to detect in experiments.

One of the main goals of the mathematical diffraction theory is to be able to deduce implications of properties of the underlying point set $\varLambda$ to the qualitative properties, such as presence, absence, density of the support, etc., of each of these three components.

3.3 Inflation tilings and Fourier cocycles

We focus on tilings that are generated by *inflation rules*, which are rules that act on a finite number of starting tiles called *prototiles*. The rule ϱ sends a prototile $\mathfrak{t}$ into a finite non-overlapping gapless union of the different prototiles and their translates.

In dimension one, the prototiles are intervals in $\mathbb{R}$, and ϱ sends an interval into a finite concatenation of intervals; see Fig. 5.

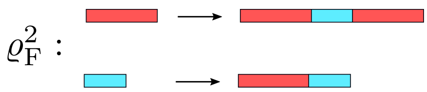


Figure 5: The squared Fibonacci inflation rule ϱ_{F}^2

The corresponding lengths of the prototiles are $|\mathfrak{t}_{\text{red}}| = \tau$ and $|\mathfrak{t}_{\text{blue}}| = 1$, where τ is the golden ratio. The inflation rule ϱ_{F}^2 sends in Fig. 5 sends each tile to a concatenation of tiles, whose total length is $\lambda = \tau^2$ times the original length of the starting prototile, hence the term inflation rule. We call λ the inflation multiplier associated to ϱ_{F}^2 . Dimension-one inflation rules are normally derived from symbolic objects called substitutions, where one requires these substitutions to be *primitive*, i.e., that they have primitive substitution matrices.

Moreover, this example also admits a *bi-infinite tiling fixed point*, which one can obtain by applying the rule infinitely many times to two red tiles joined together. We call this concatenation that generates the tiling of $\mathbb{R}$ a *seed*. The generated fixed point is given in Fig. 6.

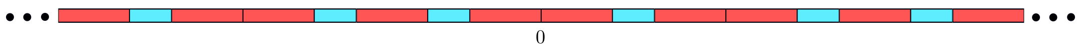


Figure 6: The tiling fixed point $\mathcal{T}$ generated by ϱ_{F}^2

From this tiling of $\mathbb{R}$, one can then get a point set $\varLambda$ by collapsing each interval into a single point, which we situate at the tile's left endpoint. This yields a point set composed of red and blue points, which is usually called a coloured point set; see Fig. 7. It is easy to see that since we started with finitely many prototiles of possibly varying lengths, the resulting point set is Delone.

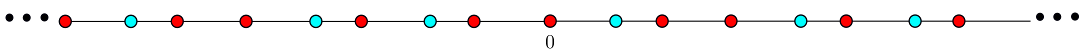


Figure 7: The point set $\varLambda$ derived from $\mathcal{T}$

To this point set, one can then get the measure ω which represents the atomic distribution, and the measure we need for diffraction analysis. It follows from general theory that this point set, as well as the tiling it came from, is aperiodic, but one can also show that it has pure point diffraction.

In higher dimensions, everything works analogously as in one dimension, except that there is the obvious additional freedom for the geometry of the prototiles. Moreover, the concept of an inflation multiplier no longer makes sense, hence we normally talk about a linear expansive map Q which satisfies

$$Q(\mathfrak{t}_i) \underset{\varrho}{\longmapsto} \bigcup_{j=1}^{n} \mathfrak{t}_j + F_{ij}$$

where F_{ij} is a finite set. A two-dimensional inflation rule is given in Fig. 8. For this example, one has $Q = \lambda^2 \mathbb{I}_2$, where $\lambda = \frac{1}{2}(5 + \sqrt{5})$.

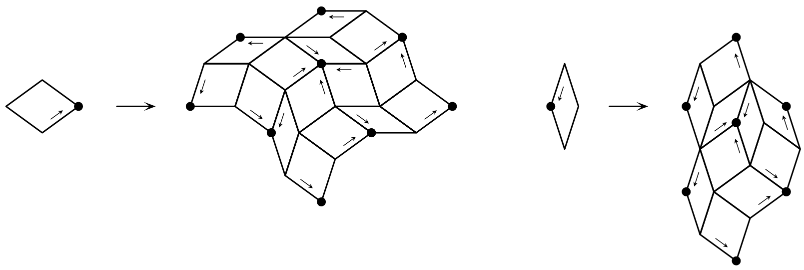


Figure 8: The inflation rule ϱ_{GLB} which generates the Godrèche–Lançon–Billard tiling in Fig. 3

To an inflation rule ϱ, both in one and in higher dimensions, one can associate an $n_a \times n_a$-matrix which encodes the positions of prototiles $\mathfrak{t}_i$ in supertiles $\varrho(\mathfrak{t}_j)$. This matrix is called the *Fourier matrix* $B(k)$ associated to ϱ. Recall that the positions of tiles are always given by their left endpoints. These positions can be seen explicitly in $B(k)$.

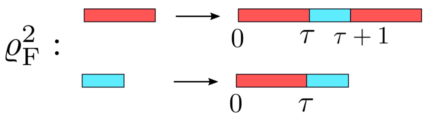


Figure 9: The squared Fibonacci inflation rule ϱ_{F}^2 with the corresponding position of tiles in supertiles

The corresponding Fourier matrix for ϱ_{F}^2 is given by

$$B(k) = \begin{pmatrix} 1 + \mathrm{e}^{2\pi \mathrm{i}(\tau+1)k} & 1 \\ \mathrm{e}^{2\pi \mathrm{i}\tau k} & \mathrm{e}^{2\pi \mathrm{i}\tau k} \end{pmatrix},$$

where k is a real parameter. For ϱ_{GLB}, one has ten prototiles (up to translation and identification), hence the matrix $B(k)$ is a 10×10 matrix, where $k \in \mathbb{R}^2$. The number of tiles gives the dimension of $B(k)$, whereas the dimension of the space where the tiling lives dictates the dimension of k.

3.4 Absence of absolutely continuous diffraction

Let $B(k)$ be the Fourier matrix associated to ϱ and let λ be the inflation multiplier. Consider the matrix cocycle whose generator is $B(k)$ with base dynamics given by $f : k \mapsto \lambda k$. The nth level cocycle is given by

$$B^{(n)}(k) = B(k)B(\lambda k) \cdots B(\lambda^{n-1}k).$$

Consider the extremal Lyapunov exponent given by

$$\chi^B(k) := \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\|,$$

which depends on the real parameter k.

The following main result gives a sufficient criterion for ruling out absolutely continuous diffraction using the Lyapunov exponent $\chi^B(k)$.

Theorem 3.4.1 ([BGäM18], 2019). *Let ϱ be a one-dimensional inflation rule, with inflation multiplier λ, and let $B(k)$ be the corresponding Fourier matrix. Under some mild assumptions, if there is an $\varepsilon > 0$ such that*

$$\chi^B(k) \leqslant \log \sqrt{\lambda} - \varepsilon$$

for Lebesgue-a.e. $k \in \mathbb{R}$, then the diffraction $\widehat{\gamma}$ of any point set arising from ϱ does not contain an absolutely continuous component.

Sketch of proof.

(1) **Fact:** $h(k)$ cannot grow exponentially as $k \to \infty$

(2) **From finiteness of prototiles:** $h(k)$ is determined by some vectors $v(k) \in \mathbb{C}^{n_a}$

(3) **From the inflation structure:** $v(k)$ satisfies some renormalisation equation involving $B(k)$

(4) If all Lyapunov exponents for $B^{(n)}(k)$ are positive, then all possible candidates for $v(k)$ also grow exponentially under $B^{(n)}(k)$

(5) $v(k) = 0$ Lebesgue-a.e. $\implies$ $h(k) = 0$ Lebesgue a.e. $\implies$ $\widehat{\gamma}_{\mathsf{ac}} = 0$

□

For higher-dimensional examples, we have the following generalisation.

Theorem 3.4.2 ([BGäM18], 2019). *Let ϱ be an inflation rule in $\mathbb{R}^d$, with expansive linear map Q, and let $B(k)$ be the corresponding Fourier matrix. Under some mild assumptions, if there is an $\varepsilon > 0$ such that*

$$\chi^B(k) \leqslant \log \sqrt{|\det Q|} - \varepsilon$$

for Lebesgue-a.e. $k \in \mathbb{R}^d$, then the diffraction $\widehat{\gamma}$ of any point set arising from ϱ does not contain an absolutely continuous component. □

3.5 Examples

3.5.1 Bijective Abelian

The following family of examples is systematically treated in [BGäM18, Sec. 4].

Example 3.5.1. Let $\mathcal{A} = \{0, 1, 2\}$ and consider ϱ_{ab} to be

$$\varrho : \begin{cases} 0 \mapsto \boxed{0}\,2\;1\,\boxed{1}\,2 \\ 1 \mapsto \boxed{1}\,0\;2\,\boxed{2}\,0 \\ 2 \mapsto \boxed{2}\,1\;0\,\boxed{0}\,1 \end{cases} .$$

This is an example of a *bijective substitution*. The zeroth and the third column are $\mathcal{C}_0 = (0, 1, 2)^{\mathrm{T}}$ and $\mathcal{C}_3 = (1, 2, 0)^{\mathrm{T}}$, respectively.

Constant-length substitutions are the simplest substitutions which give rise to inflation rules in the sense that, the inflation multiplier λ is given by the length L, and each letter gets identified to a tile of unit length. Since tile positions correspond to positions of letters in substituted words, it suffices to consider the symbolic picture in this setting. The corresponding inflation is given in Figure 10.

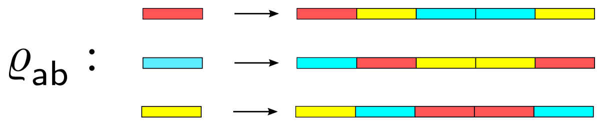


Figure 10: The inflation rule corresponding to ϱ_{ab}

Remark 3.5.2. One can consider the columns of ϱ as permutations in $S_{|\mathcal{A}|}$ via

$$\sigma_{\mathcal{C}_\ell}(0, 1, 2, \ldots, |\mathcal{A}| - 1)^{\mathrm{T}} = \mathcal{C}_\ell.$$

Definition 3.5.3. A bijective substitution ϱ is *Abelian* if the group G "generated by the columns", i.e., $G = \langle \sigma_{\mathcal{C}_\ell} \rangle$ is Abelian.

Proposition 3.5.4. *Let ϱ be Abelian. Then*

$$B(k) = \sum_{\ell=0}^{L-1} \mathrm{e}^{2\pi \mathrm{i} \ell k} D_\ell$$

where $\{D_\ell\}$ are commutative permutation matrices.

Definition 3.5.5. Given a polynomial $P(x) \in \mathbb{C}[x]$, its *logarithmic Mahler measure* is given by the mean

$$\mathfrak{m}(P) = \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i} t})| \mathrm{d}t.$$

Proposition 3.5.6. *Let ϱ be Abelian. Then the $|\mathcal{A}|$ Lyapunov exponents of $B(k)$ are given by $\chi_j(k) = \mathfrak{m}(P_j)$, for a.e. $k \in \mathbb{R}$, where $\mathfrak{m}(P_j)$ is the logarithmic Mahler measure of the polynomial*

$$P_j(x) = \sum_{\ell=0}^{L-1} \overline{\rho_j(\sigma_{\mathcal{C}_\ell})} x^\ell, \quad \textit{where } \rho_j \textit{ is the } j\textit{th irrep of } G.$$

Theorem 3.5.7. *Let ϱ be a primitive, bijective, Abelian substitution. Then, for a.e. $k \in \mathbb{R}$, all Lyapunov exponents of $B(k)$ are strictly less than $\log \sqrt{\lambda}$. Moreover, the corresponding diffraction $\widehat{\gamma}$ does not contain an absolutely continuous component.*

3.5.2 Fibonacci Tiling

The example given by the squared Fibonacci inflation in Fig. 5 is one of the classic examples of one-dimensional inflations with irrational inflation multiplier λ. For such rules, one cannot reduce the cocycle $B^{(n)}(k)$ into a cocycle on the torus $[0, 1)$, unlike the previous example where λ is always an integer because of the constant-length property. Precisely because of this, one cannot directly apply Oseledec's theorem to obtain existence and almost everywhere constancy of the exponent $\chi^B(k)$. Nevertheless, due to the quasiperiodicity of $B^{(n)}(k)$ (finiteness of the fundamental frequencies in the entries of $B(k)$), one can express $B(k)$ as a section of a periodic function over $\mathbb{T}^r$, where r is the algebraic degree of λ; compare with [BGrM18].

For ϱ_{F}^2, $\lambda = \tau^2 = \frac{3+\sqrt{5}}{2}$, whose algebraic degree is $r = 2$. For this inflation, one has the representation of the Fourier matrix as

$$B(k) = \widetilde{B}(x,y)|_{x=k,\ y=\lambda k}.$$

This allows one to consider a cocycle on $\mathbb{T}^r$ and use this cocycle to bound $\chi^B(k)$. In particular, one has the following results for general one-dimensional inflations.

Proposition 3.5.8. *Let ϱ be a primitive one-dimensional inflation. Under some mild assumptions, there exist an ergodic cocycle $\widetilde{B}$ on $\mathbb{T}^r$, such that*

$$B^{(n)}(k) = \widetilde{B}^{(n)}(x_1,\,\ldots,\,x_r)|_{x_1=k,x_2=\theta_1 k,\ldots,x_r=\theta_{r-1}k}$$

where the set $\{1,\theta_1,\ldots,\theta_{r-1}\}$ consists of rationally independent frequencies.

Using some results on averaging almost periodic functions along uniformly distributed sequences in [BHL17], one gets the following useful result.

Proposition 3.5.9 (Sequence of uniform upper bound of $\chi^B(k)$)**.** *Let ϱ be a primitive one-dimensional inflation. Under some mild assumptions,*

$$\chi^B(k) \leqslant \frac{1}{N}\int_{\mathbb{T}^r} \log\|\widetilde{B}^{(N)}(x)\|\,\mathrm{d}\mu_{\mathrm{H}}(x),$$

for a.e. $k\in\mathbb{R}$ for all $N\geqslant 1$. Here, $\widetilde{B}^{(N)}(x)$ is an ergodic cocycle on $\mathbb{T}^r$.

The previous proposition provides a sequence of uniform upper bounds for the Lyapunov exponent χ^B, which one can compute using numerical methods (quasi-Monte Carlo integration). The values for the bounds are then to be compared with $\log\sqrt{\lambda}$, and if for a particular N, the said upper bound is strictly less than $\log\sqrt{\lambda}$, one can then invoke Theorem 3.4.1 to rule out the presence of absolutely continuous diffraction.

For the squared Fibonacci, the pertinent numerical values are given in Table 3.1.

N	1	2	3	4
$\frac{1}{N}\int_{\mathbb{T}^2}\log\|\widetilde{B}^{(N)}(k)\|_{\mathrm{F}}^2$	1.5668	1.1091	0.8776	0.7409

Table 3.1: Numerical values for upper bounds for $2\chi^B(k)$ for ϱ_{F}^2. Here $\|\cdot\|_{\mathrm{F}}$ stands for the Frobenius norm.

At $N=3$, the upper bound for $2\chi^B(k)$ crosses the threshold $\log\lambda = 0.9624$, which implies that the point set coming from ϱ_{F}^2 has no absolutely continuous diffraction. Using exactly the same method, one can prove that the diffraction of the vertex set of a GLB tiling generated generated by the inflation rule ϱ_{GLB} given in Figure 8 has essentially singular continuous diffraction; see [BGäM18, Man19].

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